4.1 - *Logarithms and Their Properties*

What is a Logarithm?
We define the *common logarithm function*, or simply the *log function*, written $\log_{10} x$ or $\log x$, as follows:

If x is a positive number, log x is the exponent of 10 that gives x. In other words, if $y = \log x$ then $10^y = x$.

Example 1 on pg. 152 in Text
Rewrite the following statements using exponents instead of log.

a. $\log\ 100 = 2$

b. $\log\ 0.01 = -2$

c. $\log\ 30 = 1.477$

Example 2 on pg. 152 in Text
Rewrite the following statements using logs instead of exponents.

a. $10^5 = 100,000$

b. $10^{-4} = 0.0001$

c. $10^{0.8} = 6.3096$

Logarithms are Exponents
Logarithms are just exponents! Thinking in terms og exponents is often a good way to answer a logarithm problem.

Logarithmic and Exponential Functions are Inverses
The operation of taking a logarithm "undoes" the exponential function; the logarithm and the exponential functions are inverse functions. In particular:

For any N, $\log\left(10^N\right) = N$ and for $N > 0$, $10^{\log N} = N$

Example 4 on pg. 153 in Text
Evaluate without a calculator.

a. $\log\left(10^{8.5}\right)$

b. $10^{\log 2.7}$

c. $10^{\log(x+3)}$

Properties of Logarithms

Properties of the Common Logarithm

- By definition, $y = \log x$ means $10^y = x$.
- In particular,

$$\log 1 = 0 \quad \text{and} \quad \log 10 = 1.$$

- The functions 10^x and $\log x$ are inverses, so they "undo" each other:

$$\log(10^x) = x \quad \text{for all } x,$$
$$10^{\log x} = x \quad \text{for } x > 0.$$

- For a and b both positive and any value of t,

$$\log(ab) = \log a + \log b$$
$$\log\left(\frac{a}{b}\right) = \log a - \log b$$
$$\log(b^t) = t \cdot \log b.$$

Example 5 on pg. 154 in Text
Solve $100 \cdot 2^t = 337{,}000{,}000$ for t.

Example *(Exercise #16 on pg. 157)*
Solve $2/7 = \left(0.6\right)^{2t}$ for t.

The Natural Logarithm
When e is used as the base for exponential functions, computations are easier with the use of another logarithm function, called log base e.

For $x > 0$,

$\ln x$ is the power of e that gives x

or, in symbols,

$$\ln x = y \quad \text{means} \quad e^y = x,$$

and y is called the **natural logarithm** of x.

Just as the functions 10^x and $\log x$ are inverses, so are the functions e^x and $\ln x$.

Properties of the Natural Logarithm

- By definition, $y = \ln x$ means $x = e^y$.
- In particular,

$$\ln 1 = 0 \quad \text{and} \quad \ln e = 1.$$

- The functions e^x and $\ln x$ are inverses, so they "undo" each other:

$$\ln(e^x) = x \qquad \text{for all } x,$$
$$e^{\ln x} = x \qquad \text{for } x > 0.$$

- For a and b both positive and any value of t,

$$\ln(ab) = \ln a + \ln b$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\ln(b^t) = t \cdot \ln b.$$

Example 6 on pg. 155 in Text
Solve of x:

a. $5e^{2x} = 50$

b. $3^x = 100$

Example *(Exercise #46 on pg. 159)*
Solve for t:
$17e^{0.02t} = 18e^{0.03t}$

Example *(Not an Example in the Section*)
Solve the following equations exactly if possible for x or t:

a. $3\log(t-5) = 6$

b. $\ln x - \ln(x-1) = 1/2$

c. $2\ln(3x) + 5 = 8$

d. $e^x = 3x + 5$

e. $\ln x = -x^2$

4.2 - *Logarithms and Exponential Models*

The log function is often useful when answering questions about exponential models. Because logarithms "undo" the exponential functions, we use them to solve many exponential equations.

Example 1 on pg. 159 in Text
In Example 3 on pg. 125, (A 200 ug sample of carbon-14 decays according to the formula $Q = 200(0.886)^t$, where *t* is in thousands of years. Estimate when there is $25\mu g$ of carbon-14 left) we solved the equation $200(0.886)^t = 25$ graphically. Now solve $200(0.886)^t = 25$ using logarithms.

Example 2 on pg. 159 in Text
The US population, *P*, in millions, is currently growing according to the formula $P = 299e^{0.009t}$, where *t* is in years since 2006. When is the population predicted to reach 350 million?

Review ***Example 3 on pg. 160***

Doubling Time
Eventually, any exponentially growing quantity doubles, or increases by 100%. Since its percent growth rate is constant, the time it takes for the quantity to grow by 100% is also a constant. This time period is called the *doubling time*.

Example 4 on pg. 161 in Text

a. Find the time needed for the turtle population described by the formula $P = 175(1.145)^t$ to double its initial value.
b. How long does this population take to quadruple its initial size?

Example 5 on pg. 161 in Text
A population doubles in size every 20 years. What is its continuous growth rate?

Half-Life
An exponentially decaying quantity decreases by a factor of 2 in a fixed amount of time, called the *half-life* of the quantity.

Example 6 on pg. 162 in Text
Carbon-14 decays radioactively at a constant annual rate of 0.0121%. Show that the half-life of carbon-14 is about 5728 years.

Converting Between $Q = ab^t$ and $Q = ae^{kt}$

Any exponential function can be written in either of the two forms:

$$Q = ab^t \qquad \text{or} \qquad Q = ae^{kt}$$

If $b = e^k$, so $k = \ln b$, the two formulas represent the same function.

Example 8 on pg. 163 in Text

Convert the exponential function $P = 175(1.145)^t$ to the form $P = ae^{kt}$. Find the continuous and annual percent growth rates, assuming t is in years.

Example 9 on pg. 163 in Text

Convert the formula $Q = 7e^{0.3t}$ to the form $Q = ab^t$. Find the continuous and annual percent growth rates, assuming t is in years.

Example 11 on pg. 163 in Text

Find the continuous percent growth rate of $Q = 200(0.886)^t$, where t is in thousands of years.

4.3 - *The Logarithmic Function*

The Graph, Domain, and Range of the Common Logarithm

- The domain of log x is all positive numbers.
- The range of log x is all real numbers.
- The log function is increasing and its graph is concave down, since its rate of change is decreasing.

Graphs of the Inverse Functions $y = \log x$ and $y = 10^x$

Asymptotes

Let $y = f(x)$ be a function and let a be a finite number.

- The graph of f has a **horizontal asymptote** of $y = a$ if

$$\lim_{x\to\infty} f(x) = a \quad \textit{or} \quad \lim_{x\to-\infty} f(x) = a \quad \textit{or both.}$$

- The graph of f has a **vertical asymptote** of $x = a$ if

$$\lim_{x\to a^+} f(x) = \infty \quad \textit{or} \quad \lim_{x\to a^+} f(x) = -\infty \quad \textit{or} \quad \lim_{x\to a^-} f(x) = \infty \quad \textit{or} \quad \lim_{x\to a^-} f(x) = -\infty$$

Notice the process of finding a vertical asymptote is different from the process for finding a horizontal asymptote. Vertical asymptotes occur where the function values grow larger and larger, either positively or negatively, as x approaches a finite value. Horizontal asymptotes are determined by whether the function values approach a finite number as x takes on large positive or large negative values.

Graph of Natural Logarithm

The natural log and the common log have similar graphs.

Example 11 on pg. 163 in Text

Graph $y = \ln x$ for $0 < x < 10$.

Example *(Exercise #10 on pg. 173)*

Graph the function and label all asymptotes and intercepts. State the domain.

$y = \ln(x+1)$