Chapter 3.1 - *Introduction to the Family of Exponential Growth*

Growing at a Constant Percent Rate

Linear functions represent quantities that change at a constant rate. In this section we will look at functions that change at a constant *percent* rate, the *exponential functions*.

Population Growth

Exponential functions provide a reasonable model for many growing populations.

Radioactive Decay

Exponential functions can also model decreasing quantities. A quantity which decreases at a constant percent rate is said to be decreasing exponentially.

A General Formula for the Family of Exponential Functions

An **exponential function** has the formula $Q = f(t) = ab^t$, $b > 0$, where *a* is the initial value of $Q\ (t = 0)$ and *b*, the base, is the growth factor:

- if $b > 1$ it is an *exponential growth* example: $f(t) = 50(1.03)^t$
- if $0 < b < 1$ it is an *exponential decay* example: $f(t) = 70(.24)^t$

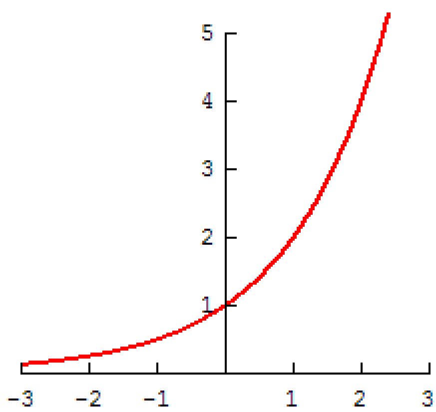
The growth factor is, $b = 1 + r$, where *r* is the decimal representation of the percent rate constant.

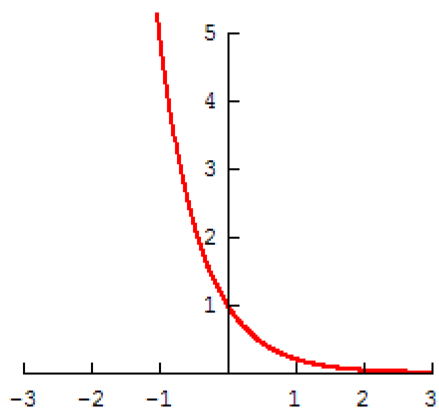
We can use the following:

- if it is an exponential growth $b = 1 + r$ and $r = b - 1$
- if it is an exponential decay $b = 1 - r$ and $r = 1 - b$

The constants *a* and *b* are called the *parameters*.

The graphs of exponential growth and decay are both *concave up*.





For a table of data that gives *y* as a function of *x* and where the change in *x* values is constant (*x* goes up by the same number) then:

- the table represents a ***linear*** function if the ***difference*** of consecutive *y*-values is constant
- the table represents a ***exponential*** function if the ***ratio*** of consecutive *y*-values is constant

$y = b + \underbrace{m + m + ... + m}_{x \text{ times}}$ repeated sums

$y = a \cdot \underbrace{b \cdot b \cdot b \cdot ... \cdot b}_{x \text{ times}}$ repeated products

Chapter 3.1 – ***Example 1 on pg. 106 in Text***

After graduation from college, you will probably be looking for a job. Suppose you are offered a job at a starting salary of $40,000 per year. To strengthen the offer, the company promises annual raises of 6% per year for at least the first five years after you are hired. Compute your salary for the first five years. Let *t* represent the number of years since the beginning of your contract. So for $t = 0$, your salary is $40,000.

$t = 0$

$S = 40,000$

$t = 1$

$S = 40,000 + 0.06(40,000) = 40,000(1 + 0.06) = 40,000(1.06) = 42,4000$

$t = 2$

$S = 42,400 + 0.06(42,400)$ or $40,000(1.06)(1.06) = 40,000(1.06)^2 = 44,944$

…and so on.

Using an exponential model:

$a = 40,000$

$r = 6\% = 0.06$

$b = 1 + r = 1 + 0.06 = 1.06$

$S = ab^t$

$S = 40,000(1.06)^t$

Chapter 3.1 – ***Example on Growth Factor*** *(Not an Example in the Section)*

What is the growth factor?

a. A city grows by 14% per year

b. A forest shrinks 7% per decade

Chapter 3.1 – ***Example Identifying Initial Value, Growth Factor, and Growth Rate*** *(Not an Example in the Section)*
Give the starting value, *a*, the growth factor, *b*, and the growth rate, *r*.

a. $Q = 2600(1.028)^t$

b. $Q = 72.9(0.89)^t$

c. $Q = 0.35(2.06)^{-2t}$

Chapter 3.1 – ***Example Describing an Exponential Function from its Formula*** *(Not an Example in the Section)*
The value, \$$V$ of an investment in year t is given by $V = 2500(1.0325)^t$. Describe the investment in words.

Chapter 3.1 – ***Example 5 on pg. 111 in Text*** *(also started as Example 3 on pg. 107)*
Carbon-14 decays at a rate of 11.4% every 1000 years. Write a formula for the quantity, Q, of a $200\mu g$ sample remaining as a function of time, t, in thousands of years.

Chapter 3.1 – ***Example 6 on pg. 111 in Text*** *(also started as Example 2 on pg. 107)*
During the early 2000's, the population of Mexico increased at a constant annual percent rate of 2%. Find a formula for P, the population of Mexico (in millions), in the year t where $t = 0$ represents the year 2000 (or where t represents the years after 2000).

Chapter 3.1 – ***Example 7 on pg. 111 in Text***
What does the formula $P = 100(1.02)^t$ predict when $t = 0$? When $t = -5$? What do these values tell you about the population of Mexico?

Chapter 3.1 – ***Example 8 on pg. 111 in Text***
If a fine starts at \$100 for the first day and then doubles daily, what is the daily percent growth rate? Find a formula for the fine as a function of t, days since the first day.

Chapter 3.2 - *Comparing Exponential and Linear Functions*

Identifying Linear and Exponential Functions from a Table

For a table of data that gives *y* as a function of *x* and in which Δx is constant:

- If the *difference* of consecutive y-values is constant, the table could represent a linear function.
- If the *ratio* of consecutive y-values is constant, the table could represent an exponential function.

Finding a Formula for an Exponential Function

To find a formula for the exponential function, we must determine the values for *a* and *b* in the formula $f(t) = ab^t$.

Modeling Linear and Exponential Growth Using Two Data Points

If we are given two data points, we can fit either a line or an exponential function to the points.

**It can be shown that an exponentially increasing quantity will, in the long run, always outpace a linearly increasing quantity.

Chapter 3.2 – *Example Identifying Linear and Exponential Functions from a Table and Finding a Formula for an Exponential Function*

x	20	25	30	35	40	45
$f(x)$	30	45	60	75	90	105
$g(x)$	1000	1200	1440	1728	2073.6	2488.32

a. Decide which function in the table above is linear and which is exponential.

b. Find a formula for the exponential function.

Chapter 3.2 – ***Example 1 on pg. 116 in Text***

At time $t = 0$ years, a species of turtle is released into a wetland. When $t = 4$, a biologist estimates there are 300 turtles in the wetland. Three years later, the biologist estimates there are 450 turtles. Let P represent the size of the turtle population in year t.

- **a.** Find a formula for P assuming linear growth. Interpret the slope and P-intercept of your formula in terms of the turtle population.

- **b.** Now find formula for P assuming exponential growth. Interpret the parameters of your formula in terms of the turtle population.

- **c.** In year $t = 12$, the biologist estimates there are 900 turtles in the wetland. What does this indicate about the two population models?

Chapter 3.2 – ***Example Modeling Linear and Exponential Growth Using Two Data Points in Functional Notation where One Point is the Initial Value***

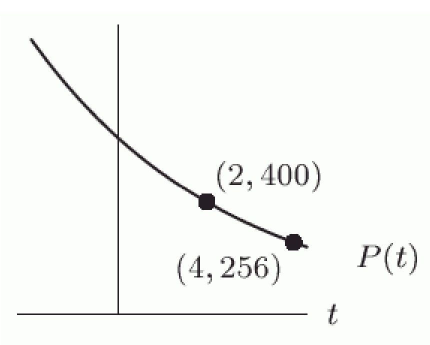
Find a formula for the function h with $h(0)=3$ and $h(-2)=8$ assuming it is:

- **a.** linear

- **b.** exponential

Chapter 3.2 – ***Example Modeling Exponential Growth Using Two Data Points from a Graph***

Find a formula for the function $P(t)$ below.



Review ***Example 2 on pg. 117*** and ***Example 3 on pg. 118 in Text.***

3.3 - *Graphs of Exponential Functions*

Graphs of the Exponential Family: The Effect of the Parameter *a*

In the formula $Q = ab^t$, the value of *a* tells us where the graph crosses the *Q*-axis, since *a* is the value of *Q* when $t = 0$.

Graphs of the Exponential Family: The Effect of the Parameter *b*

The growth factor, *b*, is called the *base* of an exponential function. Provided *a* is positive, if $b > 1$, the graph climbs when read from left to right, and if $0 < b < 1$, the graph falls when read from left to right. The growth factor, *b* affects the steepness of the graph $Q = ab^t$. For $b > 1$, the greater the value of *b*, the more rapidly the graph rises. For $0 < b < 1$, the smaller the value of *b*, the more rapidly the graph falls. In every case, however, the graph is concave up.

Horizontal Asymptotes

The horizontal line $y = k$ is a **horizontal asymptote** of a function *f*, if the function values get arbitrarily close to *k* as *x* gets large (either positively or negatively or both). We describe this behavior using the notation:

$$f(x) \to k \quad \text{as} \quad x \to \infty$$

or

$$f(x) \to k \quad \text{as} \quad x \to -\infty.$$

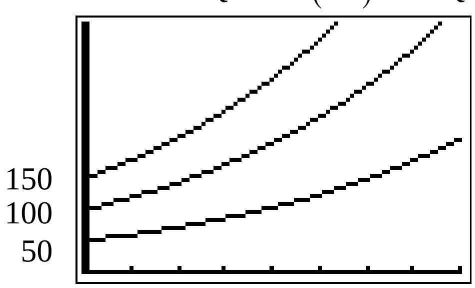
Alternatively, using limit notation, we write:

$$\lim_{x\to\infty} f(x) = k \quad \text{or} \quad \lim_{x\to-\infty} f(x) = k$$

Solving Exponential Equations Graphically

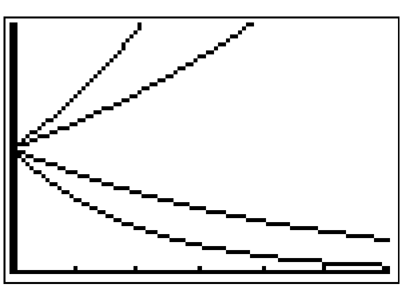
The Effect of the Parameter a

$Q = 150(1.2)^t$ $Q = 100(1.2)^t$



$Q = 50(1.2)^t$

The Effect of the Parameter b Match each formula with a function in the graph.



$Q = 50(0.6)^t$

$Q = 50(0.8)^t$

$Q = 50(1.2)^t$

$Q = 50(1.4)^t$

Chapter 3.3 – ***Example 1 on pg. 124 in Text***

A capacitor is the part of an electrical circuit that stores electric charge. The quantity of charge stored decreases exponentially with time. Stereo amplifiers provide a familiar example: When an amplifier is turned off, the display lights fade slowly because it takes time for the capacitors to discharge. If *t* is the number of seconds after the circuit is switched off, suppose that the quantity of stored charge (in micro-coulombs) is given by $Q = 200(0.9)^t$, $t \geq 0$

a. Describe in words how the stored charge changes over time.

b. What quantity of charge remains after 10 seconds? 20 seconds? 30 seconds? 1 minute? 2 minutes? 3 minutes?

c. Graph the charge over the first minute. What does the horizontal asymptote of the graph tell you about the charge?

Chapter 3.3 – ***Example 3 on pg. 125 in Text***

A 200 ug sample of carbon-14 decays according to the formula $Q = 200(0.886)^t$, where *t* is in thousands of years. Estimate when there is $25\mu g$ of carbon-14 left.

Review ***Example 2 on pg. 125***

Chapter 3.4 - *Continuous Growth and the Number e*

The Number *e*

The number e = 2.71828182… is often used for the base, b, of the exponential function. Base e is called the *natural base*.

Exponential Functions with Base *e* Represent Continuous Growth

For the exponential function $Q = ab^t$, the **continuous growth rate**, k, is given by solving $e^k = b$. Then,

$$Q = ae^{kt}.$$

If a is positive,

- If $k > 0$, then Q is increasing.
- If $k < 0$, then Q is decreasing.

Chapter 3.4 – *Example 2 on pg. 132 in Text*

A population increases from 7.3 million at a continuous rate of 2.2% per year. Write a formula for the population, and estimate graphically when the population reaches 10 million.

Chapter 3.4 – *Example 3 on pg. 132 in Text*

Caffeine leaves the body at a continuous rate of 17% per hour. How much caffeine is left in the body 8 hours after drinking a cup of coffee containing 100 mg of caffeine?

If $P = P_0(1.07)^t$, with t in years, P is growing at an *annual* rate of 7%.

If $P = P_0e^{0.07t}$, with t in years, P is growing at a *continuous* rate of 7% per year.

Since $e^{0.07} = 1.0725...$ we can rewrite $P_0e^{0.07t} = P_0(1.0725)^t$ and that would mean that $b = 1.0725$ and so $r = .0725 = 7.25\%$.

A 7% continuous rate and a 7.25% annual rate generate the same increases in P. We say that these two rates are equivalent.

Chapter 3.4 – *Example 4 on pg. 133 in Text*

If a bank offered interest at a 2.323% continuous rate, find the equivalent annual rate.

Chapter 3.5 – *Compound Interest*

Compound Interest

If interest at an annual rate of r is compounded n times a year, then $\frac{r}{n}$ times the current balance is added n times a year. Therefore, with an initial deposit of \$$P$, the balance t years later is

$$B = P\left(1+\frac{r}{n}\right)^{nt}$$

Note that r is the nominal rate; for example, $r = 0.05$ if the annual rate is 5%.

Continuous Compounding and the Number *e*

If interest on an initial deposit of \$$P$ is *compounded continuously* at an annual rate r, the balance t years later can be calculated using the formula

$$B = Pe^{rt}$$

Again, r is the nominal rate, and, for example, $r = 0.06$ when the annual rate is 6%.

Chapter 3.5 – *Example 1 on pg. 136 in Text*

What are the nominal and effective annual rates of an account paying 12% interest:

a. compounded annually

b. compounded monthly

Chapter 3.5 – *Example 2 on pg. 136 in Text*

What is the effective annual rate of an account that pays interest at the nominal rate of 6% per year,

a. compounded daily

b. compounded continuously

Chapter 3.5 – *Example 3 on pg. 138 in Text*

Which is better: An account that pays 8% annual interest compounded quarterly or an account that pays 7.95% annual interest compounded continuously?