8.1

RADIANS AND ARC LENGTH
Definition of a Radian

An angle of 1 radian is defined to be the angle, in the counterclockwise direction, at the center of a unit circle which spans an arc of length 1.

![Diagram showing a unit circle with an angle of 1 radian, indicating a radius of length 1 and an arc length of 1.](image-url)
Relationship Between Radians and Degrees

The circumference, $C$, of a circle of radius $r$ is given by $C = 2\pi r$. In a unit circle, $r = 1$, so $C = 2\pi$. This means that the arc length spanned by a complete revolution of $360^\circ$ is $2\pi$, so $360^\circ = 2\pi$ radians. Dividing by $2\pi$ gives $1$ radian $= \frac{360^\circ}{2\pi} \approx 57.296^\circ$.

• Thus, one radian is approximately $57.296^\circ$.
• One-quarter revolution, or $90^\circ$, is equal to $\frac{\pi}{2}$ or $\pi/2$ radians.
• Since $\pi \approx 3.142$, one complete revolution is about $6.283$ radians and one-quarter revolution is about $1.571$ radians.

**Equivalences for Common Angles Measured in Degrees and Radians**

<table>
<thead>
<tr>
<th>Angle in degrees</th>
<th>$0^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
<th>$60^\circ$</th>
<th>$90^\circ$</th>
<th>$120^\circ$</th>
<th>$135^\circ$</th>
<th>$150^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
<th>$360^\circ$</th>
<th>$720^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle in radians</td>
<td>$0$</td>
<td>$\pi/6$</td>
<td>$\pi/4$</td>
<td>$\pi/3$</td>
<td>$\pi/2$</td>
<td>$2\pi/3$</td>
<td>$3\pi/4$</td>
<td>$5\pi/6$</td>
<td>$\pi$</td>
<td>$3\pi/2$</td>
<td>$2\pi$</td>
<td>$4\pi$</td>
</tr>
</tbody>
</table>
Converting Between Degrees and Radians

To convert degrees to radians, or vice versa, we use the fact that $2\pi$ radians = $360^\circ$. So

$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.296^\circ \text{ and } 1^\circ = \frac{\pi}{180} \approx 0.01745 \text{ radians.}$$

• Thus, to convert from radians to degrees, multiply the radian measure by $\frac{180^\circ}{\pi}$ radians.

• To convert from degrees to radians, multiply the degree measure by $\frac{\pi \text{ radians}}{180^\circ}$. 

Converting Between Degrees and Radians

Example 3

(a) Convert 3 radians to degrees.
(b) Convert 3 degrees to radians.

Solution:

(a) $3 \text{ radians} \times \frac{180^\circ}{\pi \text{ radians}} = \frac{540^\circ}{\pi \text{ radians}} \approx 171.887^\circ$.

(b) $3^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi \text{ radians}}{60} \approx 0.052 \text{ radians}.$

The word “radians” is often dropped, so if an angle or rotation is referred to without units, it is understood to be in radians.
The **arc length**, $s$, spanned in a circle of radius $r$ by an angle of $\theta$ in radians is

$$s = r \theta.$$
Example 6
You walk 4 miles around a circular lake. Give an angle in radians which represents your final position relative to your starting point if the radius of the lake is: (a) 1 mile (b) 3 miles

Solution:

Arc length 4 and radius 1, so angle $\theta = \frac{s}{1} = 4$ radians

Arc length 4 and radius 3, so angle $\theta = \frac{s}{r} = \frac{4}{3}$ radians
Example 7

Evaluate:  
(a) \( \cos 3.14^\circ \)  
(b) \( \cos 3.14 \)

Solution:

(a) Using a calculator in degree mode, we have \( \cos 3.14^\circ = 0.9985 \). This is reasonable, because a 3.14° angle is quite close to a 0° angle, so \( \cos 3.14^\circ \approx \cos 0^\circ = 1 \).

(b) Here, 3.14 is not an angle measured in degrees; instead we interpret it as an angle of 3.14 radians. Using a calculator in radian mode, we have \( \cos 3.14 = -0.99999873 \). This is reasonable, because 3.14 radians is extremely close to \( \pi \) radians or 180°, so \( \cos 3.14 \approx \cos \pi = -1 \).
8.2

SINUSOIDAL FUNCTIONS AND THEIR GRAPHS
Amplitude and Midline

The functions

\[ y = A \sin t + k \quad \text{and} \quad y = A \cos t + k \]

have amplitude \(|A|\) and the midline is the horizontal line \(y = k\).
Amplitude and Midline

Example 1

State the midline and amplitude of the following sinusoidal functions:

(a) \( y = 3 \sin t + 5 \)  
(b) \( y = \frac{4 - 3 \cos t}{20} \).

Solution: Rewrite (b) as \( y = \frac{4}{20} - \frac{3}{20} \cos t = 0.2 - 0.15 \cos t \)

(a) Midline: \( y = 5 \)  
Amplitude: \( 3 \)

(b) Midline: \( y = 0.2 \)  
Amplitude: \( 0.15 \)

Graph of \( y = 3 \sin t + 5 \)  
Graph of \( y = 0.2 - 0.15 \cos t \)
Period

The functions

\[ y = \sin(bt) \] and \[ y = \cos(bt) \]

have period

\[ P = \frac{2\pi}{|b|} \]
Example 3

Find possible formulas for the functions $f$ and $g$ in the graphs

**Solution:**

The graph of $f$ resembles the graph of $y = \sin t$ except that its period is $P = 4\pi$. Using $P = \frac{2\pi}{B}$ gives $4\pi = \frac{2\pi}{B}$ so $B = \frac{1}{2}$ and $f(t) = \sin(\frac{1}{2} t)$

The graph of $g$ resembles the graph of $y = \sin t$ except that its period is $P = 20$. Using $P = \frac{2\pi}{B}$ gives $20 = \frac{2\pi}{B}$ so $B = \frac{\pi}{10}$ and $g(t) = \sin(\frac{\pi}{10} t)$
The graphs of

\[ y = \sin(B(t - h)) \] and \[ y = \cos(B(t - h)) \]

are the graphs of

\[ y = \sin(Bt) \] and \[ y = \cos(Bt) \]

shifted horizontally by \( h \) units.
Example 7
Describe in words the graph of the function $g(t) = \cos (3t - \pi/4)$.

Solution:
Write the formula for $g$ in the form $\cos (B(t - h))$ by factoring 3 out from the expression $(3t - \pi/4)$ to get $g(t) = \cos (3(t - \pi/12))$. The period of $g$ is $2\pi/3$ and its graph is the graph of $f = \cos 3t$ shifted $\pi/12$ units to the right, as shown.

Horizontal shift = $\pi/12$
Period = $2\pi/3$

$g(t) = \cos (3(t - \pi/12))$
$f(t) = \cos (3t)$
Summary of Transformations

For the sinusoidal functions

\[ y = A \sin(B(t - h)) + k \] and \[ y = A \cos(B(t - h)) + k, \]

- \(|A|\) is the amplitude
- \(2\pi/|B|\) is the period
- \(h\) is the horizontal shift
- \(y = k\) is the midline
- \(|B|/(2\pi)\) is the frequency; that is, the number of cycles completed in unit time.
London Eye

135-metre (443 ft) tall. It rotates at 26 cm (10 in) per second (about 0.9 km/h or 0.6 mph) so that one revolution takes about 45 minutes.
Singapore Flyer

Each of the 28 air-conditioned Capsules is capable of holding 28 passengers, and a complete rotation of the wheel takes Approximately 37 minutes. Constructed in 2005–2008. Described as an observation wheel, that reaches 42 stories high, with a total height of 165 m (541 ft)
Example 7

Use the sinusoidal function \( f(t) = A \sin(B(t - h)) + k \) to represent your height above ground at time \( t \) while riding the London Eye Ferris wheel.

**Solution:**

The diameter of the Ferris wheel is 450 feet, so the midline is \( k = 225 \) and the amplitude, \( A \), is also 225. The period of the Ferris wheel is 30 minutes, so \( B = \pi / 15 \). Because we reach \( y = 225 \) (the 3 o’clock position) when \( t = 7.5 \), the horizontal shift is \( h = 7.5 \), so the Ferris wheel height is:

\[ f(t) = 225 \sin\left(\frac{\pi}{15} (t - 7.5)\right) + 225. \]
Phase Shift

For sinusoidal functions written in the following form, $\phi$ is the phase shift:

$$y = A \sin(Bt + \phi) \text{ and } y = A \cos(Bt + \phi).$$

In the Ferris wheel height function,

$$f(t) = 225 \sin(\pi/15 (t - 7.5)) + 225,$$

rewriting the function in the above form

$$f(t) = 225 \sin(\pi/15 t - \pi/2) + 225,$$

the phase shift is $\pi/2$. 
Example 8
(a) In the figure, by what fraction of a period is the graph of $g(t)$ shifted from the graph of $f(t)$?
(b) What is the phase shift?

Solution:
(a) The period of $f(t)$ is the length of the interval from A to B. The graph of $g(t)$ appears to be shifted $1/4$ period to the right.
(b) The phase shift is $1/4 \times (2\pi) = \pi/2$. 

8.3

TRIGONOMETRIC FUNCTIONS: RELATIONSHIPS AND GRAPHS
Relationships Between the Graphs of Sine and Cosine

Example 1
Use the fact that the graphs of sine and cosine are horizontal shifts of each other to find relationships between the sine and cosine functions.

Solution:
The figure suggests that the graph of \( y = \sin t \) is the graph of \( y = \cos t \) shifted right by \( \pi/2 \) radians (or by 90°). Likewise, the graph of \( y = \cos t \) is the graph of \( y = \sin t \) shifted left by \( \pi/2 \) radians. Thus,

\[
\sin t = \cos(t - \pi/2) \\
\cos t = \sin(t + \pi/2)
\]

The graph of \( y = \sin t \) can be obtained by shifting the graph of \( y = \cos t \) to the right by \( \pi/2 \) radians.
Example 2
Use the symmetry of the graph of cosine to obtain the following relationships: (a) \( \sin t = \cos(\pi/2 - t) \) (b) \( \cos t = \sin(\pi/2 - t) \)

Solution:
(a) Since cosine has even symmetry, we can factor out \(-1\) to write

\[
\cos(\pi/2 - t) = \cos(-(t - \pi/2)) = \cos(t - \pi/2)
\]

And from Example 1, \( \cos(t - \pi/2) = \sin t \). Putting these two facts together gives us what we wanted to show:

\[
\cos(\pi/2 - t) = \sin t.
\]

(b) Again from Example 1, we know \( \cos t = \sin (t + \pi/2) \). But since cosine has even symmetry, we can replace \( t \) with \(-t\), giving us what we wanted to show:

\[
\cos t = \cos(-t) = \sin(-t + \pi/2) = \sin(\pi/2 - t).
\]
Converting from radians to degrees, we reinterpret the results of Example 2 in terms of the triangles in the figure. We see that:

\[ \sin \theta = \frac{a}{c} = \cos \phi \quad \text{and} \quad \cos \theta = \frac{b}{c} = \sin \phi \]

Since the angles in any triangle add to 180°, we have:

\[ \theta + \phi + 90^\circ = 180^\circ \quad \text{so} \quad \theta = 90^\circ - \phi \quad \text{and} \quad \phi = 90^\circ - \theta. \]

We conclude, for \( \theta \) in degrees, that

\[ \sin \theta = \cos (90^\circ - \theta) \quad \text{and} \quad \cos \theta = \sin (90^\circ - \theta), \]

which corresponds to the results for radians we found in Example 2.
Relationships Involving The Tangent Function

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{for} \quad \cos \theta \neq 0 \]
Relationship Between The Graphs of the Sine, Cosine, and Tangent Functions

The figure shows a graph of \( \sin t \) and \( \cos t \) along with a graph of \( \tan t = \frac{\sin t}{\cos t} \).

- Since a fraction equals zero where its numerator is zero, the tangent function has the same zeros as the sine function, at \( t = -2\pi, -\pi, 0, \pi, 2\pi, \ldots \). The graphs of \( y = \tan t \) and \( y = \sin t \) cross the \( t \)-axis at the same points.

- Since a fraction equals one where its numerator equals its denominator, \( \tan t = 1 \) where \( \sin t = \cos t \), which happens where the graphs of sine and cosine intersect, at \( t = -3\pi/4, \pi/4, 5\pi/4, \ldots \).

- Since a fraction is undefined where its denominator is zero, \( \tan t \) is undefined where \( \cos t = 0 \), which happens at \( t = -3\pi/2, -\pi/2, \pi/2, 3\pi/2, \ldots \), where the graph of \( y = \cos t \) crosses the \( t \)-axis. The graph of \( y = \tan t \) has vertical asymptotes at these points.
\[ y = \sin t \]
\[ y = \cos t \]
\[ y = \tan t \]
With calculator
Relationships Involving Reciprocals of the Trigonometric Functions

The reciprocals of the trigonometric functions are given special names. Where the denominators are not equal to zero, we define

\[
\text{secant } \theta = \sec \theta = \frac{1}{\cos \theta}.
\]

\[
\text{cosecant } \theta = \csc \theta = \frac{1}{\sin \theta}.
\]

\[
\text{cotangent } \theta = \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}.
\]

Example 3
Use a graph of $g(\theta) = \cos \theta$ to explain the shape of the graph of $f(\theta) = \sec \theta$.

Solution:
The figure shows the graphs of $\cos \theta$ and $\sec \theta$. In the first quadrant $\cos \theta$ decreases from 1 to 0, so $\sec \theta$ increases from 1 toward $+\infty$. The values of $\cos \theta$ are negative in the second quadrant and decrease from 0 to $-1$, so the values of $\sec \theta$ increase from $-\infty$ to $-1$. The graph of $y = \cos \theta$ is symmetric about the vertical line $\theta = \pi$, so the graph of $f(\theta) = \sec \theta$ is symmetric about the same line. Since $\sec \theta$ is undefined wherever $\cos \theta = 0$, the graph of $f(\theta) = \sec \theta$ has vertical asymptotes at $\theta = \pi/2$ and $\theta = 3\pi/2$. 
Relationships Between the Graphs of Secant and Cosine

The graphs of \( y = \csc \theta \) and \( y = \cot \theta \) are obtained in a similar fashion from the graphs of \( y = \sin \theta \) and \( y = \tan \theta \), respectively.

Plots of \( y = \csc \theta \) and \( y = \sin \theta \)
Plots of \( y = \cot \theta \) and \( y = \tan \theta \)
The Pythagorean Identity

We now see an extremely important relationship between sine and cosine. The figure suggests that no matter what the value of $\theta$, the coordinates of the corresponding point $P$ satisfy the following condition: $x^2 + y^2 = 1$. But since $x = \cos \theta$ and $y = \sin \theta$, this means $\cos^2 \theta + \sin^2 \theta = 1$.
Summarizing the Trigonometric Relationships

• Sine and Cosine functions (transformations)
  \[
  \sin t = \cos(t - \pi/2) = \cos(\pi/2 - t) = -\sin(-t)
  \]
  \[
  \cos t = \sin(t + \pi/2) = \sin(\pi/2 - t) = \cos(-t)
  \]

• Pythagorean Identity
  \[
  \cos^2 \theta + \sin^2 \theta = 1
  \]

• Tangent and Cotangent
  \[
  \tan \theta = \frac{\cos \theta}{\sin \theta} \quad \text{and} \quad \cot \theta = \frac{1}{\tan \theta}
  \]

• Secant and Cosecant
  \[
  \sec \theta = \frac{1}{\cos \theta} \quad \text{and} \quad \csc \theta = \frac{1}{\sin \theta}
  \]
8.4

TRIGONOMETRIC EQUATIONS AND INVERSE FUNCTIONS
A rabbit population in a national park rises and falls each year. It is at its minimum of 5000 rabbits in January. By July, as the weather warms up and food grows more abundant, the population triples in size. By the following January, the population again falls to 5000 rabbits, completing the annual cycle. Use a trigonometric function to find a possible formula for $R = f(t)$, where $R$ is the size of the rabbit population as a function of $t$, the number of months since January.
sketch
Solving Trigonometric Equations Graphically

A trigonometric equation is one involving trigonometric functions. Consider, for example, the rabbit population of Example 6 on page 328:

\[ R = -5000 \cos\left(\frac{\pi}{6} t\right) + 10,000. \]

Suppose we want to know when the population reaches 12,000. We need to solve the trigonometric equation

\[ -5000 \cos\left(\frac{\pi}{6} t\right) + 10,000 = 12,000. \]

We use a graph to find approximate solutions to this trigonometric equation and see that two solutions are \( t \approx 3.786 \) and \( t \approx 8.214 \). This means the rabbit population reaches 12,000 towards the end of the month 3, April (since month 0 is January), and again near the start of month 8 (September).
Solving Trigonometric Equations Algebraically!

We can try to use algebra to find when the rabbit population reaches 12,000:

\[-5000 \cos(\pi/6 \ t) + 10,000 = 12,000 \text{ or } -5000 \cos(\pi/6 \ t) = 2000\]

\[\cos(\pi/6 \ t) = -0.4\]
Solving Trigonometric Equations Algebraically!

We can try to use algebra to find when the rabbit population reaches 12,000:

\[-5000 \cos(\pi/6 \, t) + 10,000 = 12,000\]

or

\[-5000 \cos(\pi/6 \, t) = 2000\]

\[\cos(\pi/6 \, t) = -0.4\]

Now we need to know the radian values having a cosine of \(-0.4\). For angles in a right triangle, we would use the inverse cosine function, \(\cos^{-1}\):

\[\pi/6 \, t = \cos^{-1} 0.4 \approx 1.982\] using a calculator in radian mode.

So \(t = (6/ \pi) (1.982) \approx 3.786\) (We will call this \(t_1\).)

To get the other answer, we observe that the period is 12 and the graph is symmetric about the line \(t = 6\). So \(t_2 = 12 - t_1 = 12 - 3.786 = 8.214\).
The Inverse Cosine Function (restriction)

In the figure, notice that on the part of the graph where $0 \leq t \leq \pi$ (solid blue), all possible cosine values from $-1$ to $1$ occur once and once only. The calculator uses the following rule:

$\cos^{-1}$ is the angle on the blue part of the graph in the figure whose cosine is $y$.

On page 299, we defined $\cos^{-1}$ for right triangles. We now extend the definition as follows: $\cos^{-1}$ is the angle between $0$ and $\pi$ whose cosine is $y$.

The solid portion of this graph, for $0 \leq t \leq \pi$, represents a function that has only one input value for each output value.
Inverse Cosine Function graph
(convert)
The inverse cosine function, also called the arccosine function, is written $\cos^{-1} y$ or $\arccos y$. We define $\cos^{-1} y$ as the angle between 0 and $\pi$ whose cosine is $y$.

More formally, we say that

$$ t = \cos^{-1} y \text{ provided that } y = \cos t \text{ and } 0 \leq t \leq \pi. $$

Note that for the inverse cosine function

- the domain is $-1 \leq y \leq 1$ and
- the range is $0 \leq t \leq \pi$. 
Evaluating the Inverse Cosine Function

Example 1
Evaluate (a) $\cos^{-1}(0)$  (b) $\arccos(1)$  (c) $\cos^{-1}(-1)$  (d) $(\cos(-1))^{-1}$

Solution:

(a) $\cos^{-1}(0)$ means the angle between 0 and $\pi$ whose cosine is 0. Since $\cos(\pi/2) = 0$, we have $\cos^{-1}(0) = \pi/2$.

(b) $\arccos(1)$ means the angle between 0 and $\pi$ whose cosine is 1. Since $\cos(0) = 1$, we have $\arccos(1) = 0$.

(c) $\cos^{-1}(-1)$ means the angle between 0 and $\pi$ whose cosine is $-1$. Since $\cos(\pi) = -1$, we have $\cos^{-1}(-1) = \pi$.

(d) $(\cos(-1))^{-1}$ means the reciprocal of the cosine of $-1$. Since (using a calculator) $\cos(-1) = 0.5403$, we have $(\cos(-1))^{-1} = (0.5403)^{-1} = 1.8508$. 

The Inverse Sine and Inverse Tangent Functions

The figure on the left shows that values of the sine function repeat on the interval $0 \leq t \leq \pi$. However, the interval $-\pi/2 \leq t \leq \pi/2$ includes a unique angle for each value of $\sin t$.

This interval is chosen because it is the smallest interval around $t = 0$ that includes all values of $\sin t$.

The figure on the right shows why this same interval, except for the endpoints, is also used to define the inverse tangent function.
The Inverse Sine and Inverse Tangent Functions

The inverse sine function, also called the arcsine function, is denoted by $sin^{-1} y$ or $arcsin y$. We define

$$t = sin^{-1} y \text{ provided that } y = \sin t \text{ and } -\pi/2 \leq t \leq \pi/2.$$  

The inverse sine has domain $-1 \leq y \leq 1$ and range $-\pi/2 \leq t \leq \pi/2$.

The inverse tangent function, also called the arctangent function, is denoted by $tan^{-1} y$ or $arctan y$. We define

$$t = tan^{-1} y \text{ provided that } y = \tan t \text{ and } -\pi/2 < t < \pi/2.$$  

The inverse tangent has domain $-\infty < y < \infty$ and

Range $-\pi/2 < t < \pi/2$. 
Evaluating the Inverse Sine and Inverse Tangent Functions

Example 1

Evaluate

(a) \(\sin^{-1}(1)\)  
(b) \(\arcsin(-1)\)  
(c) \(\tan^{-1}(0)\)  
(d) \(\arctan(1)\)

Solution:

(a) \(\sin^{-1}(1)\) means the angle between \(-\pi/2\) and \(\pi/2\) whose sine is 1. Since \(\sin(\pi/2) = 1\), we have \(\sin^{-1}(1) = \pi/2\).

(b) \(\arcsin(-1) = -\pi/2\) since \(\sin(-\pi/2) = -1\).

(c) \(\tan^{-1}(0) = 0\) since \(\tan 0 = 0\).

(d) \(\arctan(1) = \pi/4\) since \(\tan(\pi/4) = 1\).
Example 4
Solve \( \sin \theta = 0.9063 \) for \( 0^\circ \leq \theta \leq 360^\circ \). (convert to radian)

**Solution:**

Using our calculator in degree mode, we know one solution is given by \( \sin^{-1}(0.9063) = 65^\circ \).

Referring to the unit circle in the figure, we see that another angle on the interval \( 0^\circ \leq \theta \leq 360^\circ \) also has a sine of 0.9063. By symmetry, we see that this second angle is \( 180^\circ - 65^\circ = 115^\circ \).
Finding Other Solutions

Example 5
Solve \( \sin \theta = 0.9063 \) for \( -360^\circ \leq \theta \leq 1080^\circ \). (convert to radian)

Solution:

We know that two solutions are given by \( \theta = 65^\circ, 115^\circ \). We also know that every time \( \theta \) wraps completely around the circle (in either direction), we obtain another solution. This means that we obtain the other solutions:

\[
\begin{align*}
65^\circ + 1 \cdot 360^\circ &= 425^\circ \text{ wrap once around circle} \\
65^\circ + 2 \cdot 360^\circ &= 785^\circ \text{ wrap twice around circle} \\
65^\circ + (-1) \cdot 360^\circ &= -295^\circ. \text{ wrap once around the other way}
\end{align*}
\]

For \( \theta = 115^\circ \), this means that we have the following solutions:

\[
\begin{align*}
115^\circ + 1 \cdot 360^\circ &= 475^\circ \text{ wrap once around circle} \\
115^\circ + 2 \cdot 360^\circ &= 835^\circ \text{ wrap twice around circle} \\
115^\circ + (-1) \cdot 360^\circ &= -245^\circ. \text{ wrap once around the other way}
\end{align*}
\]

Thus, the solutions on the interval \( -360^\circ \leq \theta \leq 1080^\circ \) are:

\( \theta = -295^\circ, -245^\circ, 65^\circ, 115^\circ, 425^\circ, 475^\circ, 785^\circ, 835^\circ \).
Reference Angles

For an angle $\theta$ corresponding to the point $P$ on the unit circle, the reference angle of $\theta$ is the angle between the line joining $P$ to the origin and the nearest part of the $x$-axis. A reference angle is always between $0^\circ$ and $90^\circ$; that is, between 0 and $\pi/2$.

Angles in each quadrant whose reference angles are $\alpha$