5.1 - *Vertical and Horizontal Shifts*

Translations of a Function and Its Graph

If *y = g(x)* is a function and *k* is a constant, then the graph of
- *y = g(x) + k* is the graph of *y = g(x)* shifted vertically $|k|$ units. If *k* is positive, the shift is up; if *k* is negative, the shift is down.
- *y = g(x + k)* is the graph of *y = g(x)* shifted horizontally $|k|$ units. If *k* is positive, the shift is to the left; if *k* is negative, the shift is to the right.

A vertical or horizontal shift of the graph of a function is called a *translation* because it does not change the shape of the graph, but simply translates it to another position in the plane. Shifts or translations are the simplest examples of *transformations* of a function.

Inside and Outside Changes

Since *y = g(x+k)* involves a change to the input value, *x*, it is called an *inside change* to *g*. Similarly, since *y = g(x) + k* involves a change to the output value, *g(x)*, it is called an *outside change*. In general, an inside change in a function results in a horizontal change in its graph, whereas an outside change results in a vertical change.
For the function *Q = f(t)*, a change inside the function's parentheses can be called an "inside change" and a change outside the function's parentheses can be called an "outside change".

Example 5 on pg. 197 in Text
If $n = f(A)$ gives the number of gallons of paint needed to cover a house of area A ft^2 explain the meaning of $n = f(A+10)$ and $n = f(A)+10$ in the context of painting.

Example 9 on pg. 199 in Text
A graph of $f(x) = x^2$ is in figure 5.5 below. Define *g* by shifting the graph of *f*. Find a formula for *g* in terms of *f*.

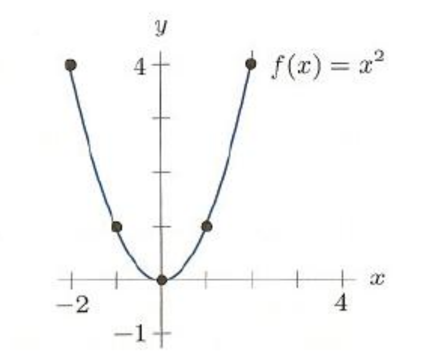


Figure 5.5: The graph of $f(x) = x^2$

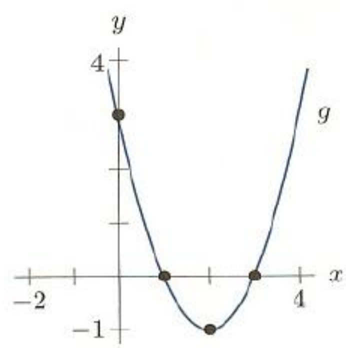


Figure 5.6: The graph of g, a transformation of f

5.1 ***Examples:***

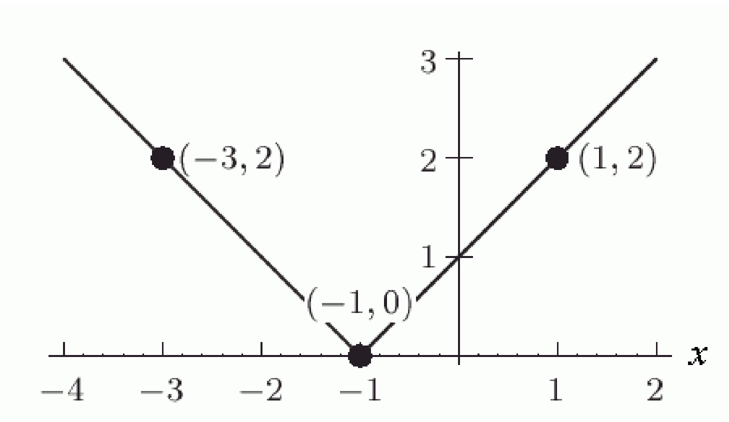
1. If $f(x)=3x^2+4x$ and $h(x)=f(x)-3$, what is $h(3)$?

2. If $r(t)=e^t$ for $t>0$, what is the formula for $s(t)=r(t-1)$?

3. If $f(x)=3x^2+3x$ what is the formula for $g(x)=f(x-4)$?

4. Let $f(x)=\ln x$ and $g(x)=\ln(x+5)$ for $x>0$. How does the graph of $g(x)$ compare to the graph of $f(x)$?

5. Given $f(x)=|x|$, what does the following figure show, give a formula?



5.2 - *Reflections and Symmetry*

In this section we consider the effect of reflecting a function's graph about the *x* or *y*-axis. A reflection about the *x*-axis corresponds to an outside change to the function's formula; a reflection about the *y*-axis corresponds to an inside change.

A Formula for Reflection

For a function f

- The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ about the *x*-axis.
- The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ about the *y*-axis.

Symmetry About the *y*-Axis
In general:

If f is a function, then f is called an **even function** if, for all values of *x* in the domain of f,

$$f(-x) = f(x).$$

The graph of f is symmetric about the *y*-axis.

Symmetry About the Origin
In general:

If f is a function, then f is called an **odd function** if, for all values of *x* in the domain of f,

$$f(-x) = -f(x).$$

The graph of f is symmetric about the origin.

Example 1 on pg. 203 in Text
Find a formula in terms of f using the table below for (*see text for the graphs*):

a. $y = g(x)$
b. $y = h(x)$
c. $y = k(x)$

x	–3	–2	–1	0	1	2	3
$f(x)$	1	2	4	8	16	32	64
$g(x)$	–1	–2	–4	–8	–16	–32	–64
$h(x)$	64	32	16	8	4	2	1
$k(x)$	–64	–32	–16	–8	–4	–2	–1

Example 2 on pg. 205 in Text

For the function $p(x)=x^2$, check algebraically that $p(-x)=p(x)$ for all *x*.

Example 3 on pg. 206 in Text

For the function $q(x)=x^3$, check algebraically that $q(-x)=-q(x)$ for all *x*.

Example 4 on pg. 207 in Text

Determine whether the following functions are symmetric about the *y*-axis, the origin, or neither.

a. $f(x)=|x|$ b. $g(x)=\dfrac{1}{x}$ c. $h(x)=-x^3-3x^2+2$

Examples:

1.

x	-3	-2	-1	0	1	2	3
$f(x)$	-4	-1	2	3	0	-3	-6
$f(-x)$							
$-f(x)$							

2. Show that the function $f(x)=x^4-x^2+7$ is even, odd, or neither <u>and</u> *explain*.

3. If $m(n)=n^2+n-3$, give a *formula* <u>and</u> *explain* the transformation $y=-m(-n)-4$

5.3 - *Vertical Stretches and Compressions*

Formula for Vertical Stretch or Compression

In general:

If f is a function and k is a constant, then the graph of $y = k \cdot f(x)$ is the graph of $y = f(x)$

- Vertically stretched by a factor of *k*, if $k > 1$.
- Vertically compressed by a factor of *k*, if $0 < k < 1$.
- Vertically stretched or compressed by a factor $|k|$ and reflected across the *x*-axis, if $k < 0$.

Stretch Factors and Average Rates of Change

Stretching or compressing a function vertically does not change the intervals on which the function increases or decreases. However, the average rate of change of a function, visible in the steepness of the graph, is altered by a vertical stretch or compression.

In general:

If $g(x) = k \cdot f(x)$, then on any interval:

Average rate of change of g = k · (Average rate of change of f).

Example 3 on pg. 215 in Text

Consider the function $y = f(x) = x^2$, graph the function $g(x) = -\frac{1}{2} f(x+3) - 1$.

Examples:

1. The following table gives values for a function f. Fill in the blanks of the table for which you have sufficient information.

x	–3	–2	–1	0	1	2	3
$f(x)$	–10	–5	4	1	0	–2	–8
$f(x+1)$							
$f(x)+3$							
$-f(x)$							
$f(-x)-2$							
$3f(x)$							

2. A carpenter currently builds k chairs per week at a cost of $f(k)$. What do the following expressions represent?

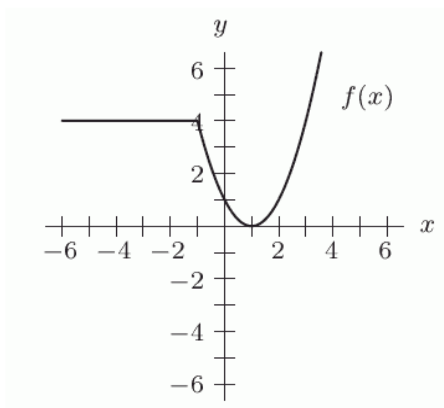
a. $f(k+10)$

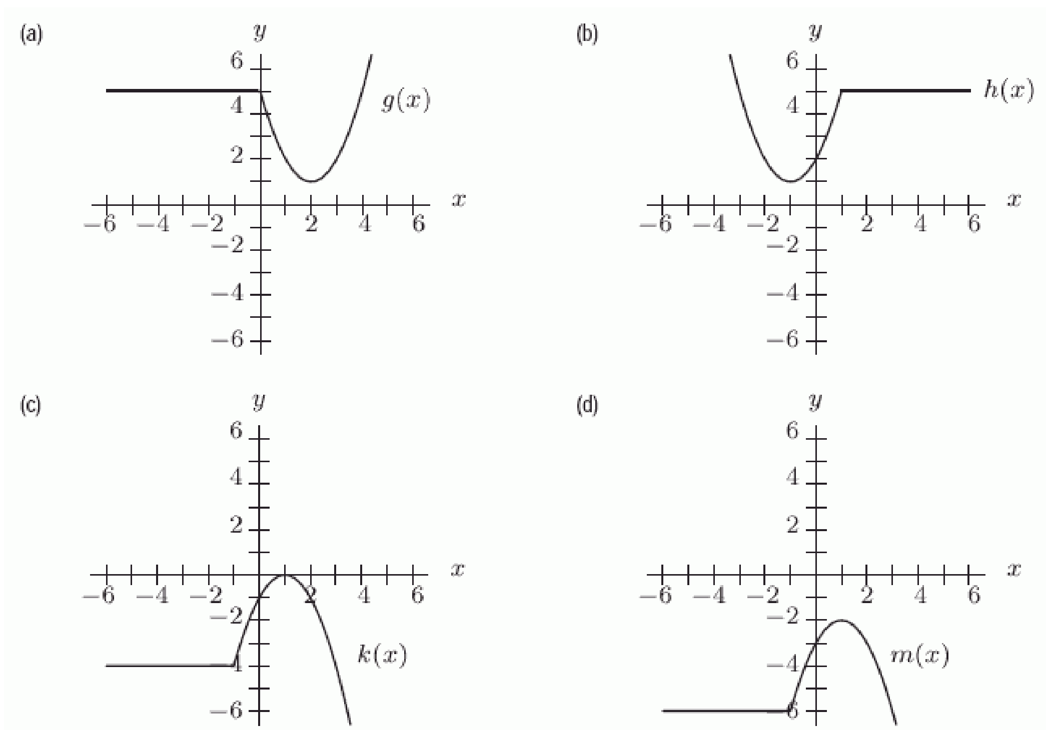
b. $f(k)+10$

c. $2f(k)$

d. $f(2k)$

3. Using the graph of $f(x)$, write formulas for functions in (a) to (d) and explain the transformation.





5.4 - *Horizontal Stretches and Compressions*

Formula for Horizontal Stretch or Compression

In general:

If f is a function and k a positive constant, then the graph of $y = f(k \cdot x)$ is the graph of f

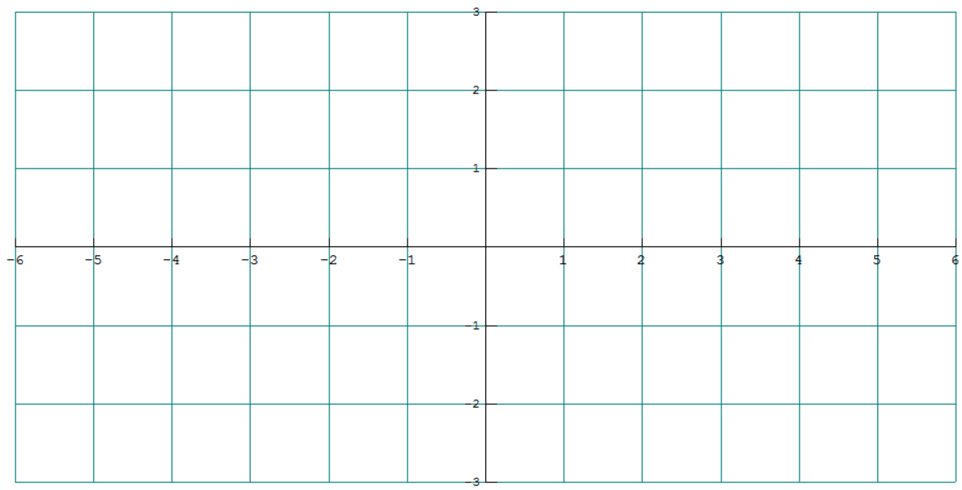
- Horizontally compressed by a factor of $1/k$ if $k > 1$.
- Horizontally stretched by a factor of $1/k$ if $k < 1$.
- If $k < 0$, then the graph of $y = f(k \cdot x)$ also involves a horizontal reflection about the *y*-axis.

Example 1 on pg. 221 in Text

The values of $f(x)$ are in the table, see the text for the graph. Make a table and a graph of the function $g(x) = f\left(\frac{1}{2}x\right)$.

x	$f(x)$
–3	0
–2	2
–1	0
0	–1
1	0
2	–1
3	1

x	$g(x)$

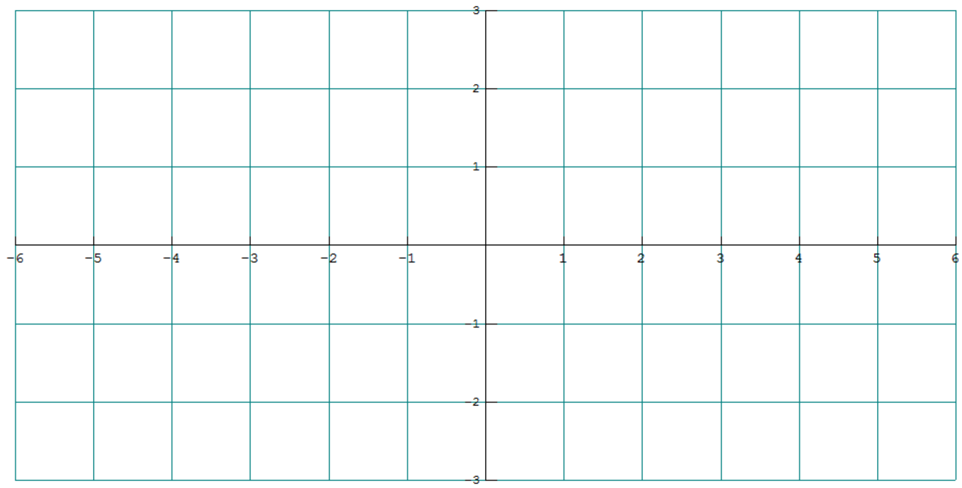


Example 2 on pg. 222 in Text

Let $f(x)$ be the function in ***Example 1***. Make a table and a graph for the function $h(x)=f(2x)$.

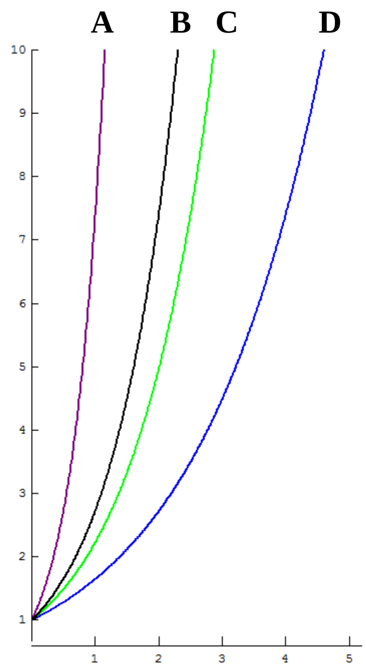
x	$f(x)$
-3	0
-2	2
-1	0
0	-1
1	0
2	-1
3	1

x	$h(x)$



Example 3 on pg. 222 in Text

Match the functions $f(t)=e^{t}$, $g(t)=e^{0.5t}$, $h(t)=e^{0.8t}$, $j(t)=e^{2t}$ with the graph below.



5.5 - *The Family of Quadratic Functions*

The graph of a quadratic function is called a *parabola*; its maximum (if the parabola opens downward) or minimum (if the parabola opens upward) is the vertex.

The Vertex of a Parabola

The graph of the function $f(x) = x^2$ is a parabola with vertex at the origin. All other functions in the quadratic family turn out to be transformations of this function.

In general, the graph of $g(x) = a(x-h)^2 + k$ is obtained from the graph of $f(x) = x^2$ by shifting horizontally $|h|$ units, stretching vertically by a factor of a (and reflecting about the x-axis if $a < 0$), and shifting vertically $|k|$ units. In the process, the vertex is shifted from (0, 0) to the point (h, k).

The graph of the function is symmetrical about a vertical line through the vertex, called the *axis of symmetry*.

In general:

The **standard form** for a **quadratic function** is

$y = ax^2 + bx + c$, where a, b, c are constants, $a \neq 0$.

The **vertex form** is

$y = a(x-h)^2 + k$, where a, h, k are constants, $a \neq 0$.

The graph of a quadratic function is called a **parabola**. The parabola

- Has vertex (h, k)
- Has axis of symmetry $x = h$
- Opens upward if $a > 0$ or downward if $a < 0$

The **factored form** is

$y = a(x-m)(x-n)$, where $x = m$ and $x = n$ are the x - intercepts and a is a constant, $a \neq 0$.

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Finding a Formula from a Graph

If we know the vertex of a quadratic function and one other point, we can use the vertex form to find its formula.