6.1 - Introduction to Periodic Functions

Periodic Functions: Period, Midline, and Amplitude
In general:

A function \( f \) is **periodic** if its values repeat at regular intervals. Graphically, this means that if the graph of \( f \) is shifted horizontally by \( c \) units, the new graph is identical to the original.

In function notation, periodic means that, for all \( t \) in the domain of \( f \),

\[
f(t + c) = f(t).
\]

The smallest positive constant \( c \) for which this relationship holds for all values of \( t \) is called the **period** of \( f \).

- **The midline** of a periodic function is the horizontal line midway between the function’s maximum and minimum values.

  **Midline:** \( y = \frac{\text{maximum} + \text{minimum}}{2} \)

- **The amplitude** is the vertical distance between the function’s maximum (or minimum) value and the midline.

  **Amplitude:** \( A = \text{maximum} - \text{midline} \) or \( A = \frac{\text{maximum} - \text{minimum}}{2} \)

World’s Largest Ferris Wheel

*Example on pg. 244 to 247 in Text*

The “London Eye” is the world’s largest ferris wheel which measures 450 feet in diameter, and carries up to 800 passengers in 32 capsules. It turns continuously, completing a single rotation once every 30 minutes. This is slow enough for people to hop on and off while it turns.

Suppose you hop on this ferris wheel at \( t = 0 \) and ride it for two full turns. The wheel is turning in a counter-clockwise direction and let \( f(t) \) be your height above the ground, measured in feet as a function of \( t \), the number of minutes you have been riding.

Find some values of \( f(t) \) and complete the table.

<table>
<thead>
<tr>
<th>( t ) minutes</th>
<th>\</th>
<th>\</th>
<th>\</th>
<th>\</th>
<th>\</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) ) feet</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
<td>\</td>
</tr>
</tbody>
</table>
Graph your ride below:

Let’s go around two more times, fill in the table below:

<table>
<thead>
<tr>
<th>$t$ minutes</th>
<th>0</th>
<th>7.5</th>
<th>15</th>
<th>22.5</th>
<th>30</th>
<th>37.5</th>
<th>45</th>
<th>52.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$ feet</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t$ minutes</td>
<td>60</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(t)$ feet</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The *period* of this function is _______, the *midline* of this function is _______ and the *amplitude* of this function is _______.

Graph this function below labeling the period, midline and amplitude:
Example determining if a function is periodic
Are the functions below periodic and if so state the period?

a.  

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(t)</td>
<td>-1</td>
<td>3</td>
<td>10</td>
<td>6</td>
<td>-1</td>
<td>3</td>
<td>10</td>
<td>6</td>
<td>-1</td>
</tr>
</tbody>
</table>

b.  

Example describing your ride on a ferris wheel given a graph
For the graph below, describe the height, \( h = f(t) \), above the ground of the ferris wheel, where \( h \) is in meters and \( t \) is in minutes. You board the wheel before \( t = 0 \). Determine the diameter of the wheel, at what height above the ground you board the wheel, how long it takes to complete one revolution, and the length of time the graph shows you riding the wheel. The boarding platform is level with the bottom of the wheel.

Example graphing a ride on a ferris wheel
Draw a graph of \( h = f(t) \) for the following ferris wheel. Label the period, the amplitude, and the midline. A ferris wheel is 60 meters in diameter and is boarded from a platform that is 4 meters above the ground. The six o’clock position on the ferris wheel is level with the loading platform. The wheel completes one full revolution every 8 minutes. At \( t = 0 \) you are in the nine o’clock position. You then make two complete revolutions and return to the boarding platform.
6.2 - The Sine and Cosine Functions

The Unit Circle
The unit circle is the circle of radius one that is centered at the origin. Since the distance from the point \( P \) on the circle with coordinates \((x, y)\) to the origin is 1, we have \( \sqrt{x^2 + y^2} = 1 \), so squaring both sides gives the equation of the circle: \( x^2 + y^2 = 1 \). Angles can be used to locate points on the unit circle. Positive angles are measured counterclockwise from the positive \( x \)-axis; negative angles are measured clockwise.

Definition of Sine and Cosine
Suppose \( P = (x, y) \) is the point on the unit circle specified by the angle \( \theta \). We define the functions, cosine of \( \theta \), or \( \cos \theta \), and the sine of \( \theta \), or \( \sin \theta \), by the formulas:

\[
\cos \theta = x \quad \text{and} \quad \sin \theta = y
\]

In other words, \( \cos \theta \) is the \( x \)-coordinate of the point on the unit circle specified by the angle \( \theta \), and \( \sin \theta \) is the \( y \)-coordinate.
Example 2 on pg. 252 in Text
Find the values for \( \cos 90^\circ, \sin 90^\circ, \cos 180^\circ, \sin 180^\circ, \cos 210^\circ, \sin 210^\circ \)

Review Example 3 on pg. 252 in Text

Coordinates of a Point on a Circle of Radius \( r \)
Using the sine and cosine, we can find the coordinates of points on circles of any size.

The coordinates \((x, y)\) of the point \(Q\) on a circle of radius \(r\) are given by
\[
    x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.
\]

Example Finding the Coordinates of a Point
Find the coordinates of points \(P\) and \(Q\) in the figure below to three decimal places if the radius of the circle is 5.

Review Example 5 on pg. 254 in Text

Example 6 on pg. 255 in Text
The ferris wheel described in section 6.1, The London Eye, has a radius of 225 feet. Find your height above ground as a function of the angle \( \theta \) measured from the 3 o’clock position. What is your height when the angle is 60°?
6.3 - Radians

So far we have measured angles in degrees. There is another way to measure an angle, which involves arc length. This is the idea behind radians; it turns out to be very helpful in calculus.

Definition of a Radian
If the radius of a circle is fixed, (say the radius is 1), the arc length is completely determined by the angle \( \theta \).

The angle of 1 radian is defined to be the angle, in the counterclockwise direction, at the center of a unit circle which spans an arc of length 1.

The radius and arc length must be measured in the same units.

An angle of 2 radians cuts off an arc of length 2 in a unit circle; an angle of \(-0.6\) radian is measured clockwise and cuts off an arc of length 0.6. In general:

The radian measure of a positive angle is the length of the arc spanned by the angle in a unit circle. For a negative angle, the radian measure is the negative of the arc length.

Radians are dimensionless units of measurement for angles (they do not have units of length).

Relationship between Radians and Degrees

\[
1 \text{ radian} = \frac{360^\circ}{2\pi} \approx 57.296^\circ
\]

Converting between Degrees and Radians
To convert from radians to degrees, multiply the radian measure by \(180^\circ/\pi\) radians.
To convert from degrees to radians, multiply the degree measure by \(\pi\) radians/\(180^\circ\).

Example 1 on pg. 258 in Text
In which quadrant is an angle of 2 radians? An angle of 5 radians?

*Go back to the unit circle on the bottom of page 4 in this handout and add approximate radian measure to each of the angles.

Example 3 on pg. 259 in Text
a. Convert 3 radians to degrees. b. Convert 3 degrees into radians

Example 4 on pg. 259 in Text
Find the arc length spanned by an angle of 30° in a circle of radius 1 meter.
Arc Length

The arc length, \( s \), spanned in a circle of radius \( r \) by an angle of \( \theta \) radians, \( 0 \leq \theta \leq 2\pi \), is given by

\[ s = r\theta \]


Example 5 on pg. 260 in Text
What length of arc is cut off by an angle of \( 120^\circ \) on a circle of radius 12 cm?

Example 6 on pg. 261 in Text
You walk 4 miles around a circular lake. Give an angle in radians which represents your final position relative to your starting point if the radius of the lake is:

a. 1 mile  

b. 3 miles

Sine and Cosine of a Number
We have defined the sine and cosine of an angle. For any real number \( t \), we define \( \sin t \) and \( \cos t \) by interpreting \( t \) as an angle of \( t \) radians.

Try these examples:
1. Sketch the positions of the following points corresponding to each angle on a circle of radius 4 and find the coordinates of each point.
   a. \( P \) is at \( 225^\circ \)
   b. \( Q \) is at \( -180^\circ \)

2. The angle \( 300^\circ \) is equivalent to _____ radians.

3. The angle \( \frac{2\pi}{3} \) radians is equivalent to _____ \(^\circ\)

4. What is the length of an arc cut off by an angle of \( 45^\circ \) in a circle of radius 5 feet?

5. What is the angle determined by an arc length \( 3\pi \) meters on a circle of radius 21 meters? Give the angle measure in radians and in degrees.
6.4 - Graphs of the Sine and Cosine

Unit Circle

30-60-90 triangle

45-45-90 triangle

3 important radian angles:

Table of values for Sine and Cosine

<table>
<thead>
<tr>
<th>θ (degrees)</th>
<th>0</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>90</th>
<th>120</th>
<th>135</th>
<th>150</th>
<th>180</th>
<th>210</th>
<th>225</th>
<th>240</th>
<th>270</th>
<th>300</th>
<th>315</th>
<th>330</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>θ (radians)</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>cos θ</td>
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<tr>
<td>sin θ</td>
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</tbody>
</table>

Graphs of the Sine and Cosine

\[ y = \cos \theta \] \hspace{2cm} \[ y = \sin \theta \]

Period: \hspace{1cm} Amplitude: \hspace{1cm} Midline: \hspace{1cm} Period: \hspace{1cm} Amplitude: \hspace{1cm} Midline:
Finding the exact value of a trigonometric function.

**Exact value = keep the answer with the radical \(\sqrt{\text{ }}\), not a decimal approximation.**

**Example**

a. Find the exact value of \(\sin\left(\frac{5\pi}{6}\right)\)  

b. Find the exact value of \(\cos\left(\frac{9\pi}{4}\right)\)

**Graphs of the Sine and Cosine Functions**

The sine is an odd function and the cosine is an even function. Since the outputs of the sine and cosine functions are the coordinates of points on the unit circle, they lie between\(-1\) and \(1\). So the range of the sine and cosine are \(-1 \leq \sin \theta \leq 1\) and \(-1 \leq \cos \theta \leq 1\). The domain of both of these functions is all real numbers, since any angle, positive or negative, specifies a point on the unit circle.
**Example 2 on pg. 265 in Text**
Compare the graph of \( y = \sin t \) to the graphs \( y = 2 \sin t \) and \( y = 0.5 \sin t \) (I added this one) and \( y = -0.5 \sin t \).

**Example not from Text**
Compare the graph of \( y = \cos t \) to the graph \( y = -3 \cos t \)

Changes in amplitude:

<table>
<thead>
<tr>
<th>( \theta ) (radians)</th>
<th>0</th>
<th>( \pi/2 )</th>
<th>( \pi )</th>
<th>( 3\pi/2 )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \cos \theta )</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>( y = -3 \cos \theta )</td>
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<td></td>
</tr>
<tr>
<td>( y = \sin \theta )</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( y = 2 \sin \theta )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{1}{2} \sin \theta )</td>
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<td></td>
</tr>
</tbody>
</table>

Amplitudes:

\[ y = \cos \theta, \ y = -3 \cos \theta \]  
\[ y = \sin \theta, \ y = 2 \sin \theta, \ y = 0.5 \sin \theta, \ y = -0.5 \sin \theta \]
Example not from Text

Compare the graph of \( y = \cos t \) to the graph \( y = \cos t - 1 \) and compare the graph of \( y = \sin t \) to the graph \( y = \sin t + 2 \).

Vertical shift:

<table>
<thead>
<tr>
<th>( \theta ) (radians)</th>
<th>0</th>
<th>( \frac{\pi}{2} )</th>
<th>( \pi )</th>
<th>( \frac{3\pi}{2} )</th>
<th>( 2\pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \cos \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \cos \theta - 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \sin \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \sin \theta + 2)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Identify the amplitude and midline of the following functions:

- \( f(x) = 3\cos x - 5 \)
- \( f(\theta) = -2\sin x + 1 \)
- \( g(t) = -\sin t - 3 \)
- \( g(\phi) = 4\cos \phi + 2 \)

Amplitude and Midline

In general:

The functions of \( y = A\sin t \) and \( y = A\cos t \) have **amplitude** \( |A| \).

The **midline** of the functions \( y = \sin t + k \) and \( y = \cos t + k \) is the horizontal line \( y = k \).
6.5 - Sinusoidal Functions

Review:
On a calculator, graph \( y = 2 \cos t - 1 \)

Amplitude: Period: Midline:

Transformations of the sine and cosine functions are called______________________________

Review:
For \( f(x) \), the transformation \( f(kx) \) is:

Example 1 on pg. 269 in Text
Graph \( y = \sin t \) and \( y = \sin (2t) \) for \( 0 \leq t \leq 2\pi \). Describe any similarities and differences. What are their periods?

What part of the curve changes when we horizontally stretch or compress a sinusoidal function?

Many functions have periods that are different from \( 2\pi \). A tide may have a period of 12 hours, a pendulum may have a period of 3 seconds. These graphs will look like sine and cosine graphs, but with different periods.
Period
If $B > 0$ the function $y = \sin(Bt)$ resembles the function $y = \sin t$ except that it is stretched or compressed horizontally. The constant $B$ determines how many cycles the function completes on an interval of length $2\pi$. Since, for $B > 0$, the graph of $y = \sin(Bt)$ completes $B$ cycles on the interval $0 \leq \theta \leq 2\pi$, each cycle has a length $\frac{2\pi}{B}$. The period is thus $\frac{2\pi}{B}$.

In general for $B$ of any sign, we have:

The functions $y = \sin(Bt)$ and $y = \cos(Bt)$ have period $P = \frac{2\pi}{|B|}$.

The number of cycles in one unit of time is $\frac{|B|}{2\pi}$, the frequency.

Example not from Text
Graph $y = \cos t$ and $y = \cos \left(\frac{1}{4}t\right)$. Describe any similarities and differences. What are their periods?

Example 2 on pg. 270 in Text
Find possible formulas for the functions below:

Example 4 on pg. 271 in Text
Describe in words the function $y = 300 \cos(0.2\pi t) + 600$ and sketch its graph.
**MAT 111 - Pre-Calculus**  

**Chapter 6 – Trigonometric Functions**

**Horizontal Shift**

The graphs of \( y = A \sin(B(t-h)) + k \) and \( y = A \cos(B(t-h)) + k \) are the graphs of \( y = \sin(Bt) \) and \( y = \cos(Bt) \) **shifted horizontally** by \( h \) units.

**Example 5 on pg. 272 in Text**

Describe in words the function \( g(t) = \cos\left(3t - \frac{\pi}{4}\right) \). State the period and the horizontal shift comparing it to which function?

Four basic graphs to know:

- \( y = \sin x \)
- \( y = -\sin x \)
- \( y = \cos x \)
- \( y = -\cos x \)

**Summary of Transformations**

The parameters \( A, B, h, \) and \( k \) determine the graph of a transformed sine or cosine function.

For the **sinusoidal functions**

\[
y = A \sin(B(t-h)) + k \quad \text{and} \quad y = A \cos(B(t-h)) + k
\]

- \( |A| \) is the amplitude
- \( h \) is the horizontal shift
- \( \frac{2\pi}{|B|} \) is the period
- \( y = k \) is the midline
- \( \frac{|B|}{2\pi} \) is the frequency, that is, the number of cycles completed in unit time.
Phase Shift

The *phase shift* enables us to calculate the fraction of a full period that the curve has been shifted.

\[ \text{Phase shift} = \text{Fraction of period} \times 2\pi. \]

For the sinusoidal functions written in the form

\[ y = A\sin(Bt + \phi) \quad \text{and} \quad y = A\cos(Bt + \phi), \]

\( \phi \) is the *phase shift*.

What is the phase shift for *Example 5* from before, \( g(t) = \cos \left( 3t - \frac{\pi}{4} \right) \)?

The graph of \( f(t) = \cos 3t \) is shifted _______ of its period to the ________?

**Try these examples:**

1. The position, \( S \), of a piston in a 6-inch stroke in an engine is given as a function of time, \( t \), in seconds, by the formula \( S = 3\sin(250\pi t) \). What is the amplitude and period of this function?

2. State the amplitude, period, midline, horizontal, and phase shift for \( y = 4\cos(2t + 5) - 3 \).

3. Find a possible formula for each graph below:
4. Suppose you are on a ferris wheel (that turns in a counter-clockwise direction) and that your height, in meters, above the ground at time, \( t \), in minutes is given by:

\[
h(t) = 20 \cos \left( \frac{1}{4} \pi t \right) + 25
\]

a. How high above the ground are you at time \( t = 0 \)?

b. What is your position on the wheel at \( t = 0 \)? (That is, what o’clock?)

c. What is the radius of the wheel?

d. How long does one revolution take?

5. A rabbit population in a national park rises and falls each year. It is at its minimum of 5000 rabbits in January. By July, as the weather warms up and food grows more abundant, the population triples in size. By the following January, the population again falls to 5000 rabbits, completing the annual cycle. Use a trigonometric function to find a possible formula for \( R = f(t) \), where \( R \) is the size of the rabbit population as a function of \( t \), the number of months since January. Graph this function.

6. Describe the function \( y = 3\sin(2x) \): amplitude___________ period ___________

Consider the new function \( y = 3\sin(2(x - \pi/4)) \). What do you expect the new function to look like?
Review
Coordinates of a point on a circle with radius $r$ is $(r \cos \theta, r \sin \theta)$

$S = r\theta$ where $\theta$ is in radian measure.

$y = A\sin(B(t-h)) + k$ and $y = A\cos(B(t-h)) + k$

- $|A|$ is the amplitude
- $\frac{2\pi}{|B|}$ is the period \( Period = \frac{2\pi}{|B|} \)
- $h$ is the horizontal shift
- $y = k$ is the midline, so $k$ is the vertical shift
- $\frac{|B|}{2\pi}$ is the frequency – that is, the number of cycles completed in unit time

**STEPS for Writing Sinusoidal Functions:**

1. Decide: if sine or cosine by where the function begins.
   - $\triangleright$ sine function begins at its midline and then goes up
   - $\triangleright$ cosine begins at its maximum and then goes down

2. Decide: if regular graph or reflected.
   - $\triangleright$ midline then goes down, reflection of sine
   - $\triangleright$ minimum, reflection of cosine
   - $\triangleright$ $A$ is negative if reflected

3. Identify: Maximum and minimum.

4. Decide: if shifted vertically
   - $\triangleright$ $k$ is the midline which is $y = \frac{\text{maximum} + \text{minimum}}{2}$
   - $\triangleright$ $k$ is positive then shifted up or if $k$ is negative then shifted down

5. Decide: on the amplitude
   - $\triangleright$ $A = \text{maximum} - \text{midline}$ or $A = \frac{\text{maximum} - \text{minimum}}{2}$

6. Decide: the length of a complete cycle or the period, then find $B$
   - $\triangleright$ $B = \frac{2\pi}{P}$

7. Decide: if shifted horizontally
   - $\triangleright$ $h$
Chapter 6 – Trigonometric Functions
Fill in The Unit Circle

Positive:
Negative:

( , )

( , )

( , )

( , )

( , )

( , )

( , )

( , )

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