4.5

THE NUMBER *e*

The *Natural* Number *e*

An irrational number, introduced by Euler in 1727, is so important that it is given a special name, *e*. Its value is approximately e ≈ 2.71828 . . .. It is often used for the base, *b*, of the exponential function. Base *e* is called the natural base. This may seem mysterious, as what could possibly be natural about using an irrational base such as *e*? The answer is that the formulas of calculus (and in nature) are much simpler if *e* is used as the base for exponentials.

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$$B = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

Evaluate \$1 at 100% interest compounded once for one year.

Evaluate $1 at 100% interest compounded once for one year.

$$B = 1 \cdot \left(1 + \frac{1}{1}\right)^{1(1)} = 2$$

Evaluate $1 at 100% interest compounded twice for one year.

$$B = 1 \cdot \left(1 + \frac{1}{2}\right)^{2(1)} = 2.25$$

Evaluate $1 at 100% interest compounded quarterly for one year.

$$B = 1 \cdot \left(1 + \frac{1}{4}\right)^{4(1)} = 2.44140625$$

Evaluate $1 at 100% interest compounded monthly for one year.

$$B = 1 \cdot \left(1 + \frac{1}{12}\right)^{12(1)} = 2.61303529$$

Evaluate $1 at 100% interest compounded daily for one year.

$$B = 1 \cdot \left(1 + \frac{1}{365}\right)^{356(1)} = 2.714567482$$

Evaluate $1 at 100% interest compounded (∞) for one year.

limit

e

2.718281828459045235360287471352662497757247093699959...

Exponential Functions with Base e

For the exponential function $Q = a\, b^t$, the continuous growth rate, k, is given by solving $e^k = b$. Then

$$Q = a\, e^{kt}.$$

If a is positive,

- **If $k > 0$, then Q is increasing.**
- **If $k < 0$, then Q is decreasing.**

Exponential Functions with Base e

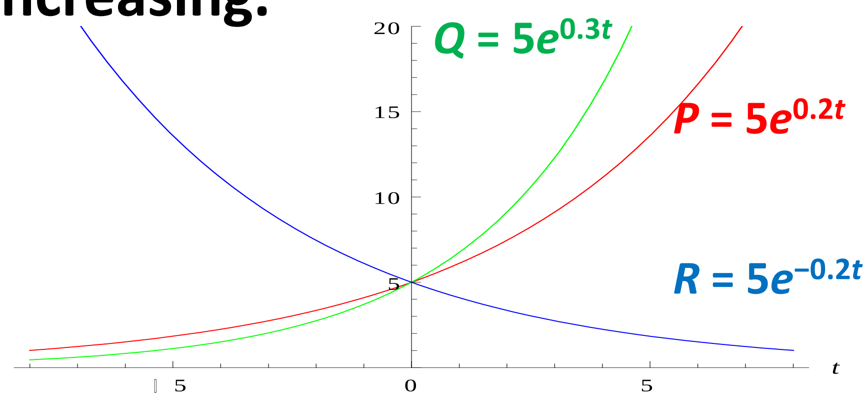
Example 1

Give the continuous growth rate of each of the following functions and graph each function:

$P = 5e^{0.2t}$, $Q = 5e^{0.3t}$, and $R = 5e^{-0.2t}$.

Solution

The function $P = 5e^{0.2t}$ has a continuous growth rate of 20%, $Q = 5e^{0.3t}$ has a continuous 30% growth rate, and $R = 5e^{-0.2t}$ has a continuous growth rate of –20%. The negative sign in the exponent tells us that R is decreasing instead of increasing.



Because $a = 5$ in all three functions, they each pass through the point (0,5). They are all concave up and have horizontal asymptote $y = 0$.

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Exponential Functions with Base e

Example 3

Caffeine leaves the body at a continuous rate of 17% per hour. How much caffeine is left in the body 8 hours after drinking a cup of coffee containing 100 mg of caffeine?

If A is the amount of caffeine in the body t hours after drinking the coffee, then

$$A = 100e^{-0.17t}.$$

Note that the continuous growth rate is −17% since A is decreasing. After 8 hours, we have

$$A = 100e^{-0.17(8)} = 25.67 \text{ mg}.$$

Connection: The Number *e* and Compound Interest

Interesting Tables

Balance after 1 year, 100% nominal interest various compounding frequencies. Compare last number to *e*.

Frequency	App.balance
1 (annually)	\$2.00
2 (semi-annually)	\$2.25
4 (quarterly)	\$2.441406
12 (monthly)	\$2.613035
365 (daily)	\$2.714567
8760 (hourly)	\$2.718127
525,600 (each minute)	\$2.718279
31,536,000 (each second)	\$2.718282

Effect of increasing the frequency of compounding, 6% nominal interest.

Compounding Frequency	Annual Growth Factor	Effective Annual Rate
Annually	1.0600000	6%
Monthly	1.0616778	6.16778%
Daily	1.0618313	6.18313%
Hourly	1.0618363	6.18363%
⋮	⋮	⋮
Continuously	$e^{0.06} \approx$ 1.0618365	6.18365%

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Connection: The Number *e* and Compound Interest

If interest on an initial deposit of $\$P$ is *compounded continuously* at a nominal rate of r per year, the balance t years later can be calculated using the formula

$$B = P e^{rt}.$$

For example, if the nominal rate is 6%, then $r = 0.06$.

Exponential Functions with Base e

Example 4

In November 2005, the Wells Fargo Bank offered interest at a 2.323% continuous yearly rate. Find the effective annual rate.

Solution:

Since $e^{0.02323} = 1.0235$, the effective annual rate is 2.35%. As expected, the effective annual rate is larger than the continuous yearly rate.

Calculate the amount of money in a bank account if $2000 is deposited for 15 years at an interest rate of:

(a) 5% annually
(b) 5% continuously per year

HW: With more terms giving a better approximation, it can be shown that

$$e = \frac{1}{1} + \frac{1}{1*2} + \frac{1}{1*2*3} + \frac{1}{1*2*3*4} + ...$$

(a) Use a calculator to sum the five terms shown.

(b) Find the sum of the first seven terms.

(c) Compare your sums with the calculator's displayed value for e (which you can find by entering e^1) and state the number of correct digits in the five and seven term sum.

(d) How many terms of the sum are needed in order to give a nine decimal digit approximation equal to the calculator's displayed value for *e*?

Suppose $1000 is deposited into an account paying interest at a nominal rate of 8% per year. Find the balance three years later if the interest is compounded

- **(a) Monthly**
- **(b) Weekly**
- **(c) Daily**
- **(d) Continuously**

If you need $25,000 six years from now, what is the minimum amount of money you need to deposit into a bank account that pays 5% annual interest, compounded:

(a) Annually (b) Monthly (c) Daily

(d) Your answers get smaller as the number of times of compounding increases. Why is this so?

Rank the following three bank deposit options from best to worst.

- Bank A: 7% compounded daily
- Bank B: 7.1 % compounded monthly
- Bank C: 7.05% compounded continuously

A sum of $850 is invested for 10 years and the interest is compounded quarterly. There is $1000 in the account at the end of 10 years. What is the nominal annual rate?

An investment grows by 3% per year for 10 years. By what percent does it increase over the 10-year period?

Are the functions exponential? If so, write the function in the form $f(t) = ab^x$

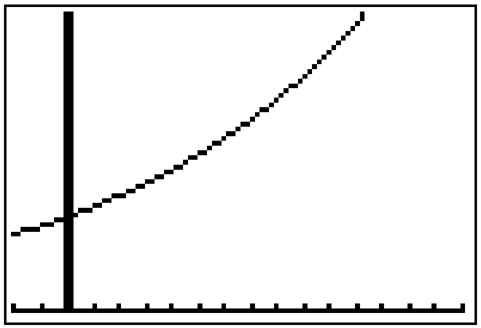
$$m(t) = (2 * 3^t)^t$$

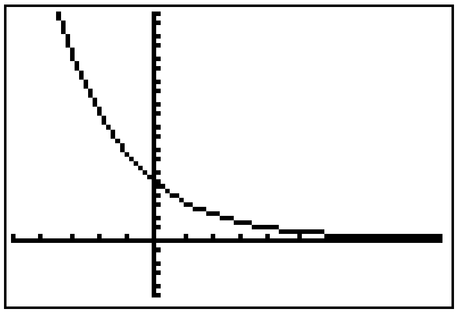
$$f(x) = \frac{3^{2x}}{4}$$

$$j(x) = 2^x 3^x$$

$$k(x) = \frac{-4}{3^x}$$

Without a calculator, match each of the following formulas to one of the graphs.

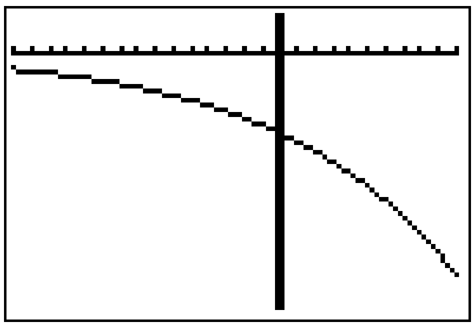


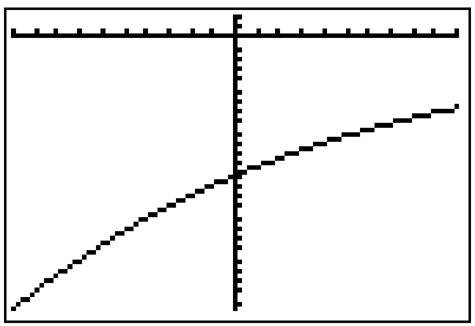


$$y = 8.3e^{-t}$$

$$y = 2.5e^{t}$$

$$y = -4e^{-t}$$





Evaluate

$$\lim_{x \to \infty} \left(257(0.93)^x \right)$$

$$\lim_{x \to -\infty} \left(15 - 5e^{3x} \right)$$

$$\lim_{x \to \infty} \left(7.2 - 2e^{3x} \right)$$

The population of a small town increases by a growth factor of 1.134 over a two-year period.

(a) By what percent does the town increase in size during the two-year period?

(b) If the town grows by the same percent each year, what is its annual percent growth rate?

*

Forty percent of a radioactive substance decays in five years. By what percent does the substance decay each year?

* **The mass, *Q*, of a sample of tritium (a radioactive isotope of hydrogen), decays at a rate of 5.626% per year. Write a function giving the mass of a 726-gram sample after a time, *t*, in years. Graph this decay function**.

A cold yam is placed in a hot oven. Newton's Law of Heating tells us that the difference between the oven's temperature and the yam's temperature decays exponentially with time. The yam's temperature is initially 0F^0, the oven's temperature is 300F^0, and the temperature difference decreases by 3% per minute. Find a formula for *Y* (*t*), the yam's temperature at time *t*.

Do Textbook Problems
Section 4.5 PAGE 163

Review problems page 168