THE TRAVELLING SALESPERSON PROBLEM:
LET US PLAN A ROAD TRIP!

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Abstract
This note offers ideas that are appropriate for exposing high school and freshman college students to topics in graph theory. A small instance of the traveling salesperson problem is introduced and students are guided to a method for solving this problem. Subsequent discussion is then used to encourage students to explore broader issues related to larger instances of this kind of problem. The pedagogical tactics described can be incorporated into a lesson plan, to suggest teaching techniques, or generally to offer suggestions to mathematics educators.

1. Objectives

The lesson “Let Us Plan a Road Trip!” described here is designed to introduce high school and freshman college students to travelling salesperson problems in a way that successfully motivates and excites students about studying a graph theory concept [1]. Students are asked the initial question: Given a small (finite) number of cities together with the cost of travelling between each pair of cities, what is the least expensive way to visit all the cities and eventually return to the starting city.

“Let Us Plan a Road Trip!” was designed as a three-part presentation. The length of each part can vary depending on the preparation of the students and the style of the instructor. The presentation can be made in as few as one meeting or as many as three sessions.

1. An introduction to tours in graphs in the form of Hamilton cycles and the closely related travelling salesperson problem.
2. A session for students to engage in interactive learning in small groups, through discussion of the content of Part 1 and extending their knowledge of the topic by means of Internet search activities.
3. A class discussion session to obtain feedback from students and their reflections on the lesson content. At this stage the students should be sufficiently prepared to make extensive use of Internet resources to locate and follow up on the vast amount of information that is available on this topic.

2. Introduction to Tours in Graphs

Four cities are suggested: New York (NY), Boston (BOS), Washington D.C. (DC), and Orlando (ORL). The students are then asked to design a road trip to start from NY, visit each of the other three cities, and then return to NY. The goal is to do this in the most efficient way. Specifically, the aim is to determine (1) the journey with the shortest total distance, and (2) the least expensive route that visits all the cities and returns to NY.

Clearly, the objectives minimize two closely related quantities: distance and cost.

To guide students toward a procedure (not necessarily the most efficient) for obtaining the optimal four-city solution, a tree diagram can be used to visually display all the possible ways to visit each city starting in and returning to NY. The six possible routes are labeled numerically 1–6 to emphasize the number of possibilities and to refer to particular route(s) later in the lesson. (See Figure 1.)
Since the problem needs to compute the total length of each possible route, a table of distances between pairs of cities is also provided, as shown in Table 1.

Table 1: Intercity distances in miles.

<table>
<thead>
<tr>
<th></th>
<th>NY (NCC)</th>
<th>BOS</th>
<th>DC</th>
<th>ORL</th>
</tr>
</thead>
<tbody>
<tr>
<td>NY (NCC)</td>
<td>183</td>
<td>209</td>
<td>943</td>
<td></td>
</tr>
<tr>
<td>BOS</td>
<td>183</td>
<td>392</td>
<td>1113</td>
<td>758</td>
</tr>
<tr>
<td>DC</td>
<td>209</td>
<td>392</td>
<td>758</td>
<td></td>
</tr>
<tr>
<td>ORL</td>
<td>943</td>
<td>1113</td>
<td>758</td>
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</tr>
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</table>

For students that have little or no prior knowledge of graph theory, time needs to be spent to introduce an appropriate type of graph for these problems; namely, a weighted complete graph. Additionally, appropriate terminology must be developed to aid with discussion of the problem and its solution: a vertex to represent each city, an edge to represent the direct route between the pair of cities it connects, an edge weight to represent the distance in miles along the route connecting two cities. For the problem under consideration, this leads to the weighted complete graph shown in Figure 2.
The next activity is to get the students to compute the total distance for each of the possible routes shown in Figure 1. Thus, for example, for Route 1 (NY–BOS–DC–ORL–NY) the total distance is

\[183 + 392 + 758 + 943 = 2276 \text{ miles}.
\]

This leads to the results shown in the second column of Table 2.

### Table 2: Total distance, gasoline consumption, and cost for each route.

<table>
<thead>
<tr>
<th>Route number</th>
<th>Total distance (miles)</th>
<th>Gasoline consumption (gallons)</th>
<th>Total cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2276</td>
<td>45.52</td>
<td>136.56</td>
</tr>
<tr>
<td>2</td>
<td>2263</td>
<td>45.26</td>
<td>135.78</td>
</tr>
<tr>
<td>3</td>
<td>2657</td>
<td>53.14</td>
<td>159.42</td>
</tr>
<tr>
<td>4</td>
<td>2263</td>
<td>45.26</td>
<td>135.78</td>
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<td>45.52</td>
<td>136.56</td>
</tr>
</tbody>
</table>

To save money a gasoline efficient car should be used to make the journey. We suggested that a hybrid car that attains 50 m.p.g. Then for Route 1 the gasoline consumption would be \(2276/50 = 45.52\) gallons. The results for each possible route are shown in the third column of Table 2. For each of the possible routes finally, to compute the cost of each trip one can use a gasoline price of $3.00/gallon, to obtain the cost for each possible route as shown in the final column of Table 2.

Table 2 shows that Routes 2 and 4 (shown boldface) are both optimal routes (shortest total distance, least gasoline consumption, smallest cost). These two routes of course result in the smallest cost ($135.78) suggesting that the trip should be planed in one of the two ways:

NY–BOS–ORL–DC–NY or
NY–DC–ORL–BOS–NY.

These two solutions, of course, represent the same cycle taken in opposite directions. The optimal route(s) can be nicely illustrated after removing unused edges from the original weighted graph as shown in Figure 3.

To close the introductory session on graphs and the travelling salesperson problem, I choose to introduce William Rowan Hamilton and his contributions to graph theory; explaining how Hamilton introduced the topic of what are now called Hamilton cycles in graphs. This topic then involved many subsequent authors to consider algorithms for locating optimal Hamilton cycles in weighted graphs, a topic known as the travelling salesperson problem. Although no efficient (that is, polynomial) algorithm for the travelling salesperson problem is known, the search for efficient algorithms that locate better suboptimal solutions continues.

### 3. Group Learning

In the second stage of this presentation, students are encouraged to engage in discussion in small groups making use of the Internet to extend their knowledge of the ideas previously introduced. It is suggested they choose a (not too large) collection of cities for themselves and calculate optimal Hamilton cycle solutions. They should be encouraged to elaborate on their reasons for selecting a particular collection of destinations,
thereby promoting their appreciation of the uses of mathematics [2]. They are asked to draw conclusions and write a brief report for later reflection.

4. Discussion

After the period of work in small groups, it is worthwhile to allow students time to reflect on successfully completed tasks and to share results with each other, explaining the reasoning they used in obtaining optimal tours and other discoveries they made while working in small groups.

It also seems useful to close the session by posing open questions that could lead students to discover additional facts about the travelling salesperson problem. This stimulates thinking and keeps new material fresh in their minds. Among the questions asked, and the student replies, were the following:

Who in real life benefits from the discoveries about the travelling salesperson problem?

One student suggested that this would be useful for musicians who are touring with their bands all over the country,

Why is there no formula or “efficient” algorithm that provides an exact solution to the travelling salesperson problem?

What does \((n - 1)!\) suggest for large \(n\)?

One student (after typing a large number for \(n\) in a TI84 graphing calculator said: “I got an overflow!

Even for relatively modest values of \(n\) the number \((n - 1)!\) Hamilton cycles in a complete graph with \(n\) vertices (cities) that start and end at a particular city, is too large for a calculator to compute.

Why did we get exact reversals in a complete weighted graph, a mirror image?

One student observed that Route 2 is the exact reverse of Route 4, it is the same city order taken backward. “Is this always the case?”

“The history of graph theory has been closely linked to applications, witness for example the importance of computer science, chemistry and electrical networks in the development of the subject. It seems reasonable to expect that applied problems will continue to play an important role, both in stimulating new graph-theoretical work and as areas where graph theory can be of practical use.” — F.S. Roberts [3]

Discussion of this kind continued for quite a while.

5. Reflection

The travelling salesperson problem has always been one of my personal favorite topics in graph theory. The fact that mathematicians have not been able to find an efficient method to solve it nor have they been able to prove that one does not exist is as fascinating to me as it is to any mathematics enthusiast. It was for this reason that I choose to explore the travelling salesperson problem with my high school students using a rather untraditional teaching style I call: learning while having fun.

This session described here could be extended by introducing brute force methods and nearest neighbor algorithms. Interactive learning could continue by assigning take home projects, making full use of the Internet, in which students can significantly increase the number of vertices (cities) and make use of approximate algorithms that (efficiently) result in suboptimal solutions that are sufficiently close to the exact solution that they can be used in practical applications.

An additional research project that could be offered (extra credit) could encourage students to explore facts about the travelling salesperson problem. Having them write a short paper on what they think makes the travelling salesperson problem impossible to solve.

Mathematics educators have the power to inspire students by being creative in their teaching styles, connecting the material they introduce to the interests of their students, and relate this material to “real world” applications.

The ideas described in this note reflect my notions about one way to accomplish this.
References


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